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RELATIONSHIPS BETWEEN BENDING SOLUTIONS OF CLASSICAL AND SHEAR DEFORMATION BEAM THEORIES

J. N. REDDY,
Department of Mechanical Engineering, Texas A & M University, College Station, TX 77843-3123, U.S.A.

C. M. WANG and K. H. LEE
Faculty of Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260

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Abstract—The exact relationships between the deflections, slopes/rotations, shear forces and bending moments of a third-order beam theory, and those of the Euler–Bernoulli theory and the Timoshenko beam theory are developed. The relationships enable one to obtain the solutions of the third-order beam theory from any known Euler–Bernoulli or Timoshenko beam theory solutions of beams, for any set of boundary conditions and transverse loads. The relationships can also be used to develop finite element models of the Timoshenko and third-order beam theories, and determine numerical solutions from the finite element model of the Euler–Bernoulli beam theory. The finite element models are free of the shear locking that is found in the conventional shear deformable finite elements. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

There are a number of beam theories that are used to represent the kinematics of deformation. To describe various beam theories, we introduce the following coordinate system. The x-coordinate is taken along the length of the beam, z-coordinate along the thickness (or height) of the beam and the y-coordinate is taken along the width of the beam. In a general beam theory, all applied loads and geometry are such that the displacements (u, v and w) along the coordinates (x, y and z) are only functions of x and z coordinates. Here we further assume that the displacement v is identically zero.

The simplest beam theory is the Euler–Bernoulli beam theory (EBT), which is based on the displacement field

$$u(x,z) = -z\frac{\mathrm{d}w_0}{\mathrm{d}x}$$
$$w(x,z) = w_0(x) \tag{1}$$

where w_0 is the transverse deflection of the point $(x, 0)$ on the midplane (i.e. $z = 0$) of the beam. The displacement field (1) implies that straight lines normal to the midplane before deformation remain straight and normal to the midsurface after deformation. The Euler–Bernoulli assumptions amount to neglecting both transverse shear and transverse normal effects, i.e. deformation is due entirely to bending and inplane stretching.

The next theory in the hierarchy of beam theories is the Timoshenko beam theory (TBT) [see Timoshenko (1921); Timoshenko and Woinowski-Krieger (1970)], which is based on the displacement field

$$u(x,z) = z\phi(x)$$
$$w(x,z) = w_0(x) \tag{2}$$

where ϕ denotes rotation of a transverse normal about the y axis. The Timoshenko beam theory relaxes the normality assumption of the Euler–Bernoulli beam theory and includes a constant state of transverse shear strain with respect to the thickness coordinate. The Timoshenko beam theory requires shear correction factors, which depend not only on the material and geometric parameters, but also on the loading and boundary conditions.

Second- and higher-order beam theories use higher-order expansions of the displacement components through the thickness of the beam. They further relax the Euler–Bernoulli hypothesis by removing the straightness assumption. In all theories the inextensibility of transverse normals can be removed by assuming that the transverse deflection also varies through the thickness. Theories higher than third order are not used because the accuracy gained is so little that the effort required to solve the equations is not justified.

A second-order theory with transverse inextensibility is based on the displacement field

$$u(x,z) = z\phi(x) + z^2\psi(x)$$
$$w(x,z) = w_0(x) \tag{3}$$

where ϕ now represents the slope $\partial u/\partial z$ at $z = 0$ of the deformed line that was straight in the undeformed beam, and ϕ and ψ together define the quadratic nature of deformed line. Similarly, a third-order beam theory [see Jemielita (1974); Levinson (1981); Bickford (1982); Reddy (1984b); Heyliger and Reddy (1988)] is based on the displacement field

$$u(x,z) = z\phi(x) + z^2\psi(x) + z^3\theta(x)$$
$$w(x,z) = w_0(x). \tag{4}$$

The following displacement field can be found in the works of Jemielita (1975), and a similar displacement field was used by Levinson (1981), Bickford (1982) and Reddy (1984b):

$$u(x,z) = z\phi(x) - \alpha z^3\left(\phi + \frac{\mathrm{d}w_0}{\mathrm{d}x}\right)$$
$$w(x,z) = w_0(x) \tag{5}$$

where $\alpha = 4/(3h^2)$. The displacement field accommodates quadratic variation of transverse shear strains (and hence stresses), and vanishing of transverse shear strain and hence, shear stress on the top and bottom planes of a beam. Thus, there is no need to use shear correction factors in a third-order theory. Levinson (1981) used a vector approach to derive the equations of equilibrium, which are essentially the same as those of the Timoshenko beam theory. Bickford (1982) and Reddy (1984b) independently derived variationally consistent equations of motion associated with the displacement field (5). Bickford's work was limited to isotropic beams, while Reddy's study considered laminated composite plates. The third-order laminated plate theory of Reddy (1984b) was specialized by Heyliger and Reddy (1988) to study linear and nonlinear bending and vibrations of isotropic beams. For other pertinent works on third-order theory of beams, the reader may consult the works of Reissner (1975), Bhimaraddi and Stevens (1984), Soldatos (1988) and Touratier (1991). The textbooks by Reddy (1984a; 1997) contain a complete review of shear deformation plate theories.

The objective of this paper is to develop exact relationships between the bending solutions of the Euler–Bernoulli beam theory and the variationally consistent third-order beam theory based on the displacement field (5). To distinguish this third-order theory from other third-order theories, hereafter this theory will be referred to as the refined beam theory (RBT). Wang (1995) developed relationships between the solutions of the Euler–Bernoulli beam theory and the Timoshenko beam theory. With the present results in hand one can obtain the exact solutions for deflections, bending moments, shear forces, of the

Timoshenko beam theory (TBT) and third-order beam theory (RBT) just by knowing the corresponding solutions of the Euler–Bernoulli beam theory, which can be found in any book on mechanics of materials.

2. EQUATIONS OF THE THIRD-ORDER BEAM THEORY

Here we develop the equations of the RBT using the displacement in eqn (5). The nonzero linear stains can be written as [see Reddy (1984a; 1997)]

$$\varepsilon_{xx} = z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)}$$
$$\gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \tag{6a}$$

where

$$\varepsilon_{xx}^{(1)} = \frac{\mathrm{d}\phi}{\mathrm{d}x}, \quad \varepsilon_{xx}^{(3)} = -\alpha\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} + \frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2}\right)$$
$$\gamma_{xz}^{(0)} = \phi + \frac{\mathrm{d}w_0}{\mathrm{d}x}, \quad \gamma_{xz}^{(2)} = -\beta\left(\phi + \frac{\mathrm{d}w_0}{\mathrm{d}x}\right) \tag{6b}$$

and

$$\alpha = \frac{4}{3h^2}, \quad \beta = 3\alpha = \frac{4}{h^2}. \tag{7}$$

The principle of virtual displacements may be used to derive the equations of equilibrium [see Reddy (1997)]. The equations of equilibrium are given by

$$\frac{\mathrm{d}\bar{Q}_x}{\mathrm{d}x} + \alpha\frac{\mathrm{d}^2 P_{xx}}{\mathrm{d}x^2} + q = 0 \tag{8}$$

$$\frac{\mathrm{d}\bar{M}_{xx}}{\mathrm{d}x} - \bar{Q}_x = 0 \tag{9}$$

where q is the transverse load,

$$\begin{Bmatrix} M_{xx} \\ P_{xx} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} z \\ z^3 \end{Bmatrix} \sigma_{xx}\,\mathrm{d}z, \quad \begin{Bmatrix} Q_x \\ R_x \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \sigma_{xz}\,\mathrm{d}z \tag{10}$$

$$\bar{M}_{xx} = M_{xx} - \alpha P_{xx}, \quad \bar{Q}_x = Q_x - \beta R_x \tag{11}$$

and (P_{xx}, R_x) denote the higher-order stress resultants.

The primary and secondary variables of the theory are:

$$\text{primary variables:} \quad w_0, \quad \frac{\mathrm{d}w_0}{\mathrm{d}x}, \quad \phi \tag{12}$$

$$\text{secondary variables:} \quad V_x, \quad P_{xx}, \quad \bar{M}_{xx} \tag{13}$$

where

$$V_x \equiv \bar{Q}_x + \alpha \frac{\mathrm{d}P_{xx}}{\mathrm{d}x}. \tag{14}$$

The specification of a primary variable constitutes a geometric boundary condition, whereas the specification of a secondary variable constitutes a force boundary condition. One should note that the present third-order theory requires the specification of both ϕ and $\mathrm{d}w_0/\mathrm{d}x$ and the effective shear force in the RBT is V_x.

The stress resultants (M_{xx}, P_{xx}, Q_x and R_x) are related to the strains by the relations

$$\begin{Bmatrix} M_{xx} \\ P_{xx} \end{Bmatrix} = \begin{bmatrix} D_{xx} & F_{xx} \\ F_{xx} & H_{xx} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{xx}^{(3)} \end{Bmatrix} \tag{15a}$$

$$\begin{Bmatrix} Q_x \\ R_x \end{Bmatrix} = \begin{bmatrix} A_{xz} & D_{xz} \\ D_{xz} & F_{xz} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^{(0)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \tag{15b}$$

where D_{xx} is the bending stiffness, A_{xz} is the shear stiffness and (F_{xx}, H_{xx}, D_{xz} and F_{xz}) are the higher-order stiffnesses:

$$\begin{aligned} D_{xx} &= \int_A E_x z^2 \,\mathrm{d}A = E_x I_{yy}^{(2)}, \quad F_{xx} = \int_A E_x z^4 \,\mathrm{d}A = E_x I_{yy}^{(4)} \\ H_{xx} &= \int_A E_x z^6 \,\mathrm{d}A = E_x I_{yy}^{(6)}, \quad A_{xz} = \int_A G_{xz} \,\mathrm{d}A = G_{xz} A \\ D_{xz} &= \int_A G_{xz} z^2 \,\mathrm{d}A = G_{xz} I_{yy}^{(2)}, \quad F_{xz} = \int_A G_{xz} z^4 \,\mathrm{d}A = G_{xz} I_{yy}^{(4)} \end{aligned} \tag{16}$$

where E_x and G_{xz} are Young's moduli and shear moduli, respectively, and $I_{yy}^{(i)}$ denotes the i-th area moment of inertia about the y-axis:

$$I_{yy}^{(i)} = \int_A (z)^i \,\mathrm{d}A \tag{17}$$

and A is the area of cross section.

It is informative to note that at a clamped edge we require $\phi = \mathrm{d}w_0/\mathrm{d}x = 0$ in the RBT. Consequently, the shear force Q_x of the EBT and TBT (and $\bar{Q}_x$) computed through constitutive equation (15b) is zero at a clamped edge in the RBT; but the effective shear force V_x [see eqn (14)] of the RBT is not zero at a clamped edge, because $\mathrm{d}P_{xx}/\mathrm{d}x$ is not zero there.

3. BENDING RELATIONSHIPS BETWEEN THE EBT, TBT AND RBT

3.1. *Summary of equations*

The bending equations of equilibrium and stress resultant-displacement relations of the three beam theories are summarized below for constant material and geometric properties:

the Euler–Bernoulli Beam Theory (EBT)

$$\frac{d^2 M_{xx}^{E}}{dx^2} = -q(x), \quad M_{xx}^{E} = -D_{xx}\frac{d^2 w_0^{E}}{dx^2}. \tag{18a,b}$$

The Timoshenko Beam Theory (TBT)

$$\frac{dM_{xx}^{T}}{dx} = Q_x^{T}, \quad \frac{dQ_x^{T}}{dx} = -q(x) \tag{19a,b}$$

$$M_{xx}^{T} = D_{xx}\frac{d\phi^{T}}{dx}, \quad Q_x^{T} = A_{xz}K_s\left(\phi^{T} + \frac{dw_0^{T}}{dx}\right). \tag{20a,b}$$

The Bickford–Reddy Beam Theory (RBT)

$$\frac{dM_{xx}^{R}}{dx} = Q_x^{R} + \alpha\frac{dP_{xx}}{dx} - \beta R_x \tag{21}$$

$$\frac{dQ_x^{R}}{dx} = -q(x) + \beta\frac{dR_x}{dx} - \alpha\frac{d^2 P_{xx}}{dx^2} \tag{22}$$

$$M_{xx}^{R} = D_{xx}\frac{d\phi^{R}}{dx} - \alpha F_{xx}\left(\frac{d\phi^{R}}{dx} + \frac{d^2 w_0^{R}}{dx^2}\right) \tag{23}$$

$$Q_x^{R} = \bar{A}_{xz}\left(\phi^{R} + \frac{dw_0^{R}}{dx}\right) \tag{24}$$

$$P_{xx} = F_{xx}\frac{d\phi^{R}}{dx} - \alpha H_{xx}\left(\frac{d\phi^{R}}{dx} + \frac{d^2 w_0^{R}}{dx^2}\right) \tag{25}$$

$$R_x = \bar{D}_{xz}\left(\phi^{R} + \frac{dw_0^{R}}{dx}\right) \tag{26}$$

where quantities with superscript "E" refer to the Euler–Bernoulli beam theory, with "T" refer to the Timoshenko beam theory (K_s denotes the shear correction factor) and quantities with "R" refer to the third-order beam theory of Reddy. We introduce the following notation for stiffness for future use:

$$\begin{aligned} \bar{D}_{xx} &= D_{xx} - \alpha F_{xx}, \quad \bar{F}_{xx} = F_{xx} - \alpha H_{xx} \\ \bar{A}_{xz} &= A_{xz} - \beta D_{xz}, \quad \bar{D}_{xz} = D_{xz} - \beta F_{xz} \\ \hat{A}_{xz} &= \bar{A}_{xz} - \beta \bar{D}_{xz}. \end{aligned} \tag{27}$$

3.2. *Relationships between the EBT and TBT solutions*

The deflection, bending moment and shear force of Timoshenko beam theory can be expressed in terms of the corresponding quantities of the Euler–Bernoulli beam theory [see Wang (1995)]. The relationships are established using the load equivalence. For example, the load–deflection relationships of the EBT and TBT are

$$-D_{xx}\frac{\mathrm{d}^4 w_0^{\mathrm{E}}}{\mathrm{d}x^4} = -q(x), \quad D_{xx}\frac{\mathrm{d}^3\phi^{\mathrm{T}}}{\mathrm{d}x^3} = -q(x), \quad A_{xz}K_s\left(\frac{\mathrm{d}\phi^{\mathrm{T}}}{\mathrm{d}x}+\frac{\mathrm{d}^2 w_0^{\mathrm{T}}}{\mathrm{d}x^2}\right) = -q(x) \tag{28}$$

where K_s is the shear correction coefficient. These equations can be used to establish the following relations [see Wang (1995)]:

$$D_{xx}w_0^{\mathrm{T}}(x) = D_{xx}w_0^{\mathrm{E}}(x)+\frac{D_{xx}}{A_{xz}K_s}M_{xx}^{\mathrm{E}}(x)+C_1\left(\frac{D_{xx}}{A_{xz}K_s}x-\frac{x^3}{6}\right)-C_2\frac{x^2}{2}-C_3x-C_4 \tag{29a}$$

$$D_{xx}\phi^{\mathrm{T}}(x) = -D_{xx}\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x}+C_1\frac{x^2}{2}+C_2x+C_3 \tag{29b}$$

$$M_{xx}^{\mathrm{T}}(x) = M_{xx}^{\mathrm{E}}(x)+C_1x+C_2 \tag{29c}$$

$$Q_x^{\mathrm{T}}(x) = Q_x^{\mathrm{E}}(x)+C_1 \tag{29d}$$

where C_1, C_2, C_3 and C_4 are constants of integration, which are to be determined using the boundary conditions of the particular beam. It can be shown that, for simply supported beams, all C_i are zero and for the clamped free (cantilever) case, all C_i except $C_4 = M_{xx}^{\mathrm{E}}(0)/D_{xx}A_{xz}K_s$ are zero.

To illustrate the evaluation of the constants of integration using boundary conditions, we consider bending of a beam of length L, clamped (or fixed) at the left end and simply supported at the right, and subjected to a uniformly distributed transverse load. The boundary conditions are as follows:

$$\text{EBT:} \quad w_0^{\mathrm{E}}(0) = w_0^{\mathrm{E}}(L) = \frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x}(0) = M_{xx}^{\mathrm{E}}(L) = 0 \tag{30}$$

$$\text{TBT:} \quad w_0^{\mathrm{T}}(0) = w_0^{\mathrm{T}}(L) = \phi^{\mathrm{T}}(0) = M_{xx}^{\mathrm{T}}(L) = 0. \tag{31}$$

These conditions yield

$$C_1 = \frac{3\Omega}{(1+3\Omega)L}M_{xx}^{\mathrm{E}}(0), \quad C_2 = -C_1L, \quad C_3 = 0, \quad C_4 = \Omega M_{xx}^{\mathrm{E}}(0)L^2 \tag{32}$$

where $\Omega = D_{xx}/(A_{xz}K_sL^2)$. For additional boundary conditions, see Wang (1995).

3.3. *Relationships between solutions of EBT and RBT*

Here we develop the relationships between the bending solutions of EBT and RBT. At the outset, we note that both the Euler–Bernoulli and Timoshenko beam theories are fourth-order theories, whereas the Bickford–Reddy beam theory is a sixth-order theory. The order referred to here is the total order of all equations of equilibrium expressed in terms of the generalized displacements. The refined beam theory is governed by a fourth-order equation in w_0 and a second-order equation in ϕ. Therefore, the relationships between the solutions of two different order theories can only be established by solving an additional second-order equation. The relationships are developed between deflections, rotations, and the stress resultants of the EBT and TBT for easy comparison between theories.

First we note that eqns (21) and (22) together yield

$$\frac{\mathrm{d}^2 M_{xx}^{\mathrm{R}}}{\mathrm{d}x^2} = -q(x). \tag{33}$$

In view of eqn (18a), i.e. equating the loads, we obtain

$$-q(x)=\frac{\mathrm{d}^2M_{xx}^{\mathrm{R}}}{\mathrm{d}x^2}=\frac{\mathrm{d}^2M_{xx}^{\mathrm{E}}}{\mathrm{d}x^2}=\frac{\mathrm{d}Q_x^{\mathrm{E}}}{\mathrm{d}x} \tag{34a}$$

or

$$\frac{\mathrm{d}M_{xx}^{\mathrm{R}}}{\mathrm{d}x}=\frac{\mathrm{d}M_{xx}^{\mathrm{E}}}{\mathrm{d}x}+C_1=Q_x^{\mathrm{E}}+C_1 \tag{34b}$$

and

$$\boxed{M_{xx}^{\mathrm{R}}(x)=M_{xx}^{\mathrm{E}}(x)+C_1x+C_2.} \tag{34c}$$

The stress resultant-displacement relationships (23)–(26) can be expressed as

$$M_{xx}^{\mathrm{R}}=\frac{\bar{D}_{xx}}{\bar{A}_{xz}}\frac{\mathrm{d}Q_x^{\mathrm{R}}}{\mathrm{d}x}-D_{xx}\frac{\mathrm{d}^2w_0^{\mathrm{R}}}{\mathrm{d}x^2} \tag{35a}$$

$$R_x=\frac{\bar{D}_{xz}}{\bar{A}_{xz}}Q_x^{\mathrm{R}} \tag{35b}$$

$$\begin{aligned}P_{xx}&=\frac{\bar{F}_{xx}}{\bar{A}_{xz}}\frac{\mathrm{d}Q_x^{\mathrm{R}}}{\mathrm{d}x}-F_{xx}\frac{\mathrm{d}^2w_0^{\mathrm{R}}}{\mathrm{d}x^2}\\&=\left(\frac{\bar{F}_{xx}}{\bar{A}_{xz}}-\frac{F_{xx}\bar{D}_{xx}}{D_{xx}\bar{A}_{xz}}\right)\frac{\mathrm{d}Q_x^{\mathrm{R}}}{\mathrm{d}x}+\left(\frac{F_{xx}}{D_{xx}}\right)M_{xx}^{\mathrm{R}}.\end{aligned} \tag{35c}$$

Replacing P_{xx} and R_x in eqn (21) with the expressions in eqns (35b, c), we obtain

$$\left(\frac{\bar{D}_{xx}}{D_{xx}}\right)\frac{\mathrm{d}M_{xx}^{\mathrm{R}}}{\mathrm{d}x}=\left(\frac{\hat{A}_{xz}}{\bar{A}_{xz}}\right)Q_x^{\mathrm{R}}-\alpha\left(\frac{F_{xx}\bar{D}_{xx}}{D_{xx}\bar{A}_{xz}}-\frac{\bar{F}_{xx}}{\bar{A}_{xz}}\right)\frac{\mathrm{d}^2Q_x^{\mathrm{R}}}{\mathrm{d}x^2}. \tag{36}$$

Using eqn (34b) and simplifying the coefficients, we arrive at

$$\alpha\left(\frac{F_{xx}\bar{D}_{xx}}{D_{xx}\bar{A}_{xz}}-\frac{\bar{F}_{xx}}{\bar{A}_{xz}}\right)\frac{\mathrm{d}^2Q_x^{\mathrm{R}}}{\mathrm{d}x^2}-\left(\frac{\hat{A}_{xz}}{\bar{A}_{xz}}\right)Q_x^{\mathrm{R}}+\left(\frac{\bar{D}_{xx}}{D_{xx}}\right)(Q_x^{\mathrm{E}}+C_1)=0. \tag{37}$$

Thus, a second-order differential equation must be solved to determine Q_x^{R} in terms of Q_x^{E}. Once Q_x^{R} is known, M_{xx}^{R}, ϕ^{R} and w_0^{R} can be determined, as will be shown shortly. The effective shear force V_x^{R} in the RBT can be computed from eqn (14):

$$\begin{aligned}V_x^{\mathrm{R}}(x)&=\bar{Q}_x^{\mathrm{R}}+\alpha\frac{\mathrm{d}P_{xx}}{\mathrm{d}x}=\frac{\mathrm{d}M_{xx}^{\mathrm{R}}}{\mathrm{d}x}\\&=Q_x^{\mathrm{E}}(x)+C_1\end{aligned} \tag{38}$$

where eqn (21) and (34b) are used to arrive at the last equality.

To determine ϕ^{R}, we use eqn (23):

$$D_{xx}\frac{d\phi^{R}}{dx} = M_{xx}^{R}+\alpha F_{xx}\left(\frac{d\phi^{R}}{dx}+\frac{d^2 w_0^{R}}{dx^2}\right)$$
$$= M_{xx}^{E}+C_1 x+C_2+\alpha\left(\frac{F_{xx}}{\bar{A}_{xz}}\right)\frac{dQ_x^{R}}{dx}$$
$$= -D_{xx}\frac{d^2 w_0^{E}}{dx^2}+C_1 x+C_2+\alpha\left(\frac{F_{xx}}{\bar{A}_{xz}}\right)\frac{dQ_x^{R}}{dx} \tag{39}$$

or

$$\boxed{D_{xx}\phi^{R}(x) = -D_{xx}\frac{dw_0^{E}}{dx}+\alpha\left(\frac{F_{xx}}{\bar{A}_{xz}}\right)Q_x^{R}+C_1\frac{x^2}{2}+C_2 x+C_3} \tag{40}$$

where eqns (18b) and (34c) are used in arriving at the last equation.

Lastly, we derive the relations between w_0^{R} and w_0^{E}. Using eqns (34) and (40), we can write

$$D_{xx}\frac{dw_0^{R}}{dx} = -D_{xx}\phi^{R}+\left(\frac{D_{xx}}{\bar{A}_{xz}}\right)Q_x^{R}$$
$$= D_{xx}\frac{dw_0^{E}}{dx}+\left(\frac{\bar{D}_{xx}}{\bar{A}_{xz}}\right)Q_x^{R}-C_1\frac{x^2}{2}-C_2 x-C_3 \tag{41}$$

and integrating with respect to x, we obtain

$$\boxed{D_{xx}w_0^{R}(x) = D_{xx}w_0^{E}(x)+\left(\frac{\bar{D}_{xx}}{\bar{A}_{xz}}\right)\left(\int^{x} Q_x^{R}(\eta)\,d\eta\right)-C_1\frac{x^3}{6}-C_2\frac{x^2}{2}-C_3 x-C_4.} \tag{42}$$

This completes the derivations of the relationships between the solutions of the EBT and RBT. The constants of integration, C_1, C_2, C_3 and C_4, appearing in eqns (34c), (40) and (42) are determined using the boundary conditions. Since there are six boundary conditions in the third-order theory [see eqns (12) and (13)], the remaining two boundary conditions are used in solving the second-order equation (37). Boundary conditions for various types of supports are defined below, consistent with the primary and secondary variables [see eqns (12) and (13)] of the theory:

$$\text{Free(F)}: \quad Q_x^{R}-\beta R_x+\alpha\frac{dP_{xx}}{dx} = 0 \quad M_{xx}^{R}-\alpha P_{xx} = 0, \quad P_{xx} = 0 \tag{43}$$

$$\text{Simply supported(S)}: \quad w_0^{R} = 0, \quad M_{xx}^{R}-\alpha P_{xx} = 0, \quad P_{xx} = 0 \tag{44}$$

$$\text{Clamped(C)}: \quad w_0^{R} = 0, \quad \phi^{R} = 0, \quad \frac{dw_0^{R}}{dx} = 0. \tag{45}$$

Since the second-order equation (37) requires boundary conditions on Q_x^{R}, we reduce the force boundary conditions in eqns (43)–(45) to one in terms of Q_x^{R}:

$$\text{Free(F)}: \quad \text{eqns(43) and(35c) imply } \frac{dQ_x^R}{dx} = 0 \tag{46a}$$

$$\text{Simply supported(S)}: \quad \text{eqn(44) implies } \frac{dQ_x^R}{dx} = 0 \tag{46b}$$

$$\text{Clamped(C)}: \quad \text{eqns(45) and(25b) imply } Q_x^R = 0. \tag{46c}$$

3.4. *Examples*

Here we present two examples to derive the solutions of the refined third-order theory using the relationships between the EBT and RBT, and the solutions of EBT.

3.4.1. *Simply supported beam.* First we consider a simply supported beam under a uniformly distributed load of intensity, q_0. For this case, the shear force Q_x^E of the EBT can be computed from the relation

$$\frac{dQ_x^E}{dx} = -q_0 \quad \text{or} \quad Q_x^E(x) = \frac{q_0}{2}(L-2x).$$

Then eqn (37) becomes

$$\frac{d^2Q_x^R}{dx^2} - \lambda^2 Q_x^R = -\mu\left[\frac{q_0}{2}(L-2x)+C_1\right] \tag{47}$$

where

$$\lambda^2 = \frac{\hat{A}_{xz}D_{xx}}{\alpha(F_{xx}\bar{D}_{xx}-\bar{F}_{xx}D_{xx})}, \quad \mu = \frac{\bar{A}_{xz}\bar{D}_{xx}}{\alpha(F_{xx}\bar{D}_{xx}-\bar{F}_{xx}D_{xx})}. \tag{48}$$

The solution to this differential equation is

$$Q_x^R(x) = C_5 \sinh\lambda x + C_6 \cosh\lambda x + \frac{\mu}{\lambda^2}\left[\frac{q_0}{2}(L-2x)+C_1\right] \tag{49}$$

where C_5 and C_6 are constants to be determined, along with C_1, C_2, C_3 and C_4, using the boundary conditions.

The boundary conditions for the problem at hand are

$$w_0^E(0) = w_0^E(L) = M_{xx}^E(0) = M_{xx}^E(L) = 0 \tag{50}$$

$$w_0^R(0) = w_0^R(L) = M_{xx}^R(0) = M_{xx}^R(L) = P_{xx}(0) = P_{xx}(L) = 0. \tag{51}$$

We note from eqn (35c) that

$$M_{xx}^R(0) = P_{xx}(0) = 0 \quad \text{imply} \quad \frac{dQ_x^R}{dx}(0) = 0 \tag{52a}$$

$$M_{xx}^R(L) = P_{xx}(L) = 0 \quad \text{imply} \quad \frac{dQ_{dx}^R}{dx}(L) = 0. \tag{52b}$$

Using the boundary conditions (50)–(52), we find that

$$C_1 = C_2 = C_3 = 0, \quad C_4 = \left(\frac{q_0\mu}{\lambda^4}\right)\left(\frac{\bar{D}_{xx}}{\bar{A}_{xz}}\right) \tag{53}$$

$$C_5 = \frac{q_0\mu}{\lambda^3}, \quad C_6 = -\frac{q_0\mu}{\lambda^3}\tanh\left(\frac{\lambda L}{2}\right) \tag{54}$$

and the solutions becomes

$$M_{xx}^{\mathrm{R}}(x) = M_{xx}^{\mathrm{E}}(x) \tag{55}$$

$$Q_x^{\mathrm{R}}(x) = \left(\frac{q_0\mu}{\lambda^3}\right)\left[\sinh\lambda x - \tanh\left(\frac{\lambda L}{2}\right)\cosh\lambda x + \frac{\lambda}{2}(L-2x)\right] \tag{56}$$

$$\phi^{\mathrm{R}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} + \left(\frac{\alpha F_{xx}}{\bar{A}_{xz}D_{xx}}\right)Q_x^{\mathrm{R}}(x) \tag{57}$$

$$w_0^{\mathrm{R}}(x) = w_0^{\mathrm{E}}(x) + \left(\frac{q_0\mu}{\lambda^4}\right)\left(\frac{\bar{D}_{xx}}{\bar{A}_{xz}D_{xx}}\right)\left[\cosh\lambda x - \tanh\left(\frac{\lambda L}{2}\right)\sinh\lambda x + \frac{\lambda^2}{2}(Lx - x^2) - 1\right] \tag{58}$$

where $M_{xx}^{\mathrm{E}} = q_0(Lx - x^2)/2$ for the problem at hand. For the same boundary conditions, the Timoshenko beam solution is given by

$$w_0^{\mathrm{T}}(x) = w_0^{\mathrm{E}}(x) + \left(\frac{1}{A_{xz}K_s}\right)M_{xx}^{\mathrm{E}}(x). \tag{59}$$

The Euler–Bernoulli beam solution for the deflection is given by

$$w_0^{\mathrm{E}}(x) = \frac{q_0L^4}{24D_{xx}}\left[\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right]. \tag{60}$$

For a rectangular cross section beam, it can be shown that

$$\frac{\bar{D}_{xx}\bar{D}_{xx}}{\bar{A}_{xz}D_{xx}D_{xx}} = \frac{6}{5A_{xz}}, \quad \frac{\bar{D}_{xx}}{\bar{A}_{xz}D_{xx}} = \frac{6}{5A_{xz}}. \tag{61}$$

A close examination of eqn (58) shows that the RBT solution has an effective shear coefficient, based on the coefficient in the expression for $w_0^{\mathrm{R}}(x)$, of $K_s = 5/6$. Of course, the refined third-order beam theory does not require a shear correction factor. Also, the shear correction factor for the Timoshenko beam theory can be obtained, e.g. by comparing the maximum deflections of the TBT and RBT.

3.4.2. *Cantilever (C–F) beam.* For a cantilever beam under uniformly distributed load of intensity, q_0, the shear force Q_x^{E} is given by

$$Q_x^{\mathrm{E}}(x) = q_0(L - x). \tag{62}$$

The general solution of eqn (37) with Q_x^{E} as defined in eqn (62) is

$$Q_x^{\mathrm{R}}(x) = C_5 \sinh \lambda x + C_6 \cosh \lambda x + \frac{\mu}{\lambda^2}[q_0(L-x)+C_1]. \tag{63}$$

The boundary conditions for the cantilever beam are

$$w_0^{\mathrm{E}}(0) = \frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x}(0) = Q_x^{\mathrm{E}}(L) = M_{xx}^{\mathrm{E}}(L) = 0 \tag{64}$$

$$w_0^{\mathrm{R}}(0) = \frac{\mathrm{d}w_0^{\mathrm{R}}}{\mathrm{d}x}(0) = \phi^{\mathrm{R}}(0) = \frac{\mathrm{d}M_{xx}^{\mathrm{R}}}{\mathrm{d}x}(L) = M_{xx}^{\mathrm{R}}(L) = P_{xx}(L) = 0. \tag{65}$$

We note from eqn (35c) that

$$M_{xx}^{\mathrm{R}}(L) = P_{xx}(L) = 0 \quad \text{imply} \quad \frac{\mathrm{d}Q_x^{\mathrm{R}}}{\mathrm{d}x}(L) = 0 \tag{66}$$

and from eqns (40) and (42)

$$\frac{\mathrm{d}w_0^{\mathrm{R}}}{\mathrm{d}x}(0) = \phi^{\mathrm{R}}(0) = \frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x}(0) = 0 \quad \text{imply} \quad Q_x^{\mathrm{R}}(0) = 0. \tag{67}$$

Although $Q_x^{\mathrm{R}}(0)$ obtained from the constitutive relations is zero at the clamped edge, the effective shear force of the theory, V_x, at $x = 0$ is indeed not zero. It is given by eqn (38).

Using the boundary conditions (64)–(67), we obtain the following expressions for the constants:

$$C_1 = C_2 = C_3 = 0, \quad C_4 = \left(\frac{\bar{D}_{xx}}{\bar{A}_{xz}}\right)\left(\frac{q_0\mu}{\lambda^4}\right)\left(\frac{1+\lambda L \sinh \lambda L}{\cosh \lambda L}\right) \tag{68}$$

$$C_5 = \frac{q_0\mu}{\lambda^3}\left(\frac{1+\lambda L \sinh \lambda L}{\cosh \lambda L}\right), \quad C_6 = -\frac{q_0 L\mu}{\lambda^2} \tag{69}$$

and the solution becomes

$$M_{xx}^{\mathrm{R}}(x) = M_{xx}^{\mathrm{E}}(x) \tag{70}$$

$$Q_x^{\mathrm{R}}(x) = \left(\frac{q_0\mu}{\lambda^3 \cosh \lambda L}\right)\left(\sinh \lambda x - \lambda L \cosh \lambda(L-x)\right) + \frac{q_0\mu}{\lambda^2}(L-x) \tag{71}$$

$$\phi^{\mathrm{R}}(x) = -\frac{\mathrm{d}w_0^{\mathrm{E}}}{\mathrm{d}x} + \left(\frac{\alpha F_{xx}}{\bar{A}_{xz} D_{xx}}\right) Q_x^{\mathrm{R}}(x) \tag{72}$$

$$\begin{aligned} w_0^{\mathrm{R}}(x) = w_0^{\mathrm{E}}(x) &+ \left(\frac{q_0\mu}{\lambda^4 \cosh \lambda L}\right)\left(\frac{\bar{D}_{xx}}{\bar{A}_{xz} D_{xx}}\right)(\cosh \lambda x + \lambda L \sinh \lambda(L-x)) \\ &+ \left(\frac{\bar{D}_{xx}}{\bar{A}_{xz} D_{xx}}\right)\frac{q_0\mu}{2\lambda^2}(2Lx - x^2) - \left(\frac{\bar{D}_{xx}}{D_{xx}\bar{A}_{xz}}\right)\left(\frac{q_0\mu}{\lambda^4}\right)\left(\frac{1+\lambda L \sinh \lambda L}{\cosh \lambda L}\right). \end{aligned} \tag{73}$$

The Timoshenko and Euler–Bernoulli beam deflections are given by

$$w_0^{\mathrm{T}}(x) = w_0^{\mathrm{E}}(x) + \frac{1}{A_{xz}K_s}(M_{xx}^{\mathrm{E}}(x) - M_{xx}^{\mathrm{E}}(0)) \tag{74}$$

$$w_0^{\mathrm{E}}(x) = \frac{q_0 L^4}{24 D_{xx}}\left[6\left(\frac{x}{L}\right)^2 - 4\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right]. \tag{75}$$

From eqns (71)–(75) it is clear that, once again, the effective shear correction factor of the RBT is $K_s = 5/6$.

4. CONCLUSIONS

In this paper exact relationships between the bending solutions of the Euler–Bernoulli beam theory (EBT) and the refined third-order beam theory (RBT) are developed. Since the RBT is a sixth-order theory, and the EBT is a fourth-order theory, the exact relationships between deflections, slopes, moments and shear forces of the two theories can only be developed by solving an additional second-order differential equation. Upon having the solution of this equation, the exact relationships between the solutions of the two theories can be established.

The relationships presented in this paper can be used to generate bending solutions of the third-order beam theory of Reddy whenever the Euler–Bernoulli beam solutions are available. Since solutions of the Euler–Bernoulli beam theory are available in most textbooks on mechanics of materials for a variety of boundary conditions, the correspondence presented herein between various theories makes it easier to compute the solutions of the TBT and RBT directly from the known Euler–Bernoulli beam solutions. It is also possible to develop finite element models of the TBT and RBT using the finite element model of EBT. The stiffness matrix of the shear deformable elements are also 4×4 for the pure bending case and the finite elements are free from shear locking phenomenon [see Reddy (1997)] experienced by the conventional shear deformable finite elements.

The present work can be easily extended to symmetrically laminated beams. Indeed, the relationships developed herein hold for symmetrical laminated beams in which the Poisson effect is neglected and the transverse deflection is assumed to be only a function of x [see Reddy (1997) for details]. The only difference lies in the calculation of beam stiffness, D_{xx}, A_{xz}, etc. which depend on individual layer stiffnesses and thickness.

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